Received: 17 February 2023 | Revised: 5 July 2023 | Accepted: 9 July 2023

DOI: 10.1002/mma.9575

RESEARCH ARTICLE

WILEY

# Survey of new applications of geometric algebra

**Eckhard Hitzer**[1] | **Manos Kamarianakis**[2] | **George Papagiannakis**[2] | **Petr Vašík**[3]

[1]College of Liberal Arts, International Christian University, Tokyo, Japan

[2]Department of Computer Science, University of Crete, Crete, Greece

[3]Faculty of Mechanical Engineering, Brno University of Technology, Brno, Czech Republic

**Correspondence**
Eckhard Hitzer, International Christian University, Osawa 3-10-2, 181-8585 Mitaka, Japan.
Email: hitzer@icu.ac.jp

Communicated by: P. Vašík

**Present address**
International Christian University, Osawa 3-10-2, 181-8585 Mitaka, Japan

This survey introduces 101 new publications on applications of Clifford's geometric algebras (GAs) newly published during 2022 (until mid-January 2023). The selection of papers is based on a comprehensive search with Dimensions.ai, followed by detailed screening and clustering. Readers will learn about the use of GA for mathematics, computation, surface representations, geometry, image, and signal processing, computing and software, quantum computing, data processing, neural networks, medical science, physics, electric engineering, control and robotics.

**KEYWORDS**

Clifford geometric algebra, computing, control, data processing, electric engineering, geometry, image and signal processing, mathematics, medical science, neural networks, physics, quantum computing, robotics, software, surface representations

**MSC CLASSIFICATION**

11E88, 15A66, 15-02, 68T01

## 1 | INTRODUCTION

Applications[1] of Clifford's geometric algebras (GAs) (Clifford algebras) are quickly increasing in numbers and diversity. In order to provide an up-to-date survey of the latest applications in mid-January of 2023, we used the Dimensions.ai search engine for the years of 2022 and 2023 with keywords *Geometric Algebra* and found 121 publications (articles, preprints, books, and book chapters) with these keywords in title or abstract. After checking each item for its appropriateness, we selected 101 of them for this survey. As survey authors, we ourselves made quite a few *discoveries* of novel applications we were not aware of so far. We hope readers will have similar moments of surprise. One reason is that Dimensions.ai is not in any kind of theme— or information bubble—but delivers all results available.

We note that GA has become popularly used in applications dealing with geometry. It allows to reformulate and redefine problems involving geometry in a highly intuitive and general way. GA was defined thanks to the work of W. K. Clifford [1] to unify and generalize Grassmann algebra [2] and W.R. Hamilton's quaternions [3] into a universal algebraic framework by adding the inner product to H. G. Grassmann's outer product. One of the GAs that is often applied is conformal geometric algebra (CGA). It became better known through Hestenes et al. [4], is well described and illustrated in Dorst et al. [5], and in a brief illustrated form in Hitzer et al. [6]. For standard references on GA, we refer to the following textbooks: earlier studies [5, 7, 8]. A brief introduction for engineers can be found in Hitzer [9], while a compact definition of GA is given in Falcao and Malonek [10]; see also previous research [9, 11].

[1]This paper is subject to the Creative Peace License, https://gaupdate.wordpress.com/2011/12/14/the-creative-peace-license-14-dec-2011/, accessed 17 Feb. 2023.

  

Following Hitzer [9], any Clifford GA $Cl(V)$ is generated from an inner-product[2] vector space $(V, a\cdot b\ :\ a, b \in V \mapsto \mathbb{R})$ by Clifford's geometric product setting[3] the geometric product[4] of any vector with itself equal to the inner product: $aa = a \cdot a$. We indeed have the *universal property* [12, 13] that any isometry[5] from the vector space $V$ into an inner-product algebra[6] $\mathcal{A}$ over the field[7] $\mathbb{K}$ can be uniquely extended to an isometry[8] from the Clifford algebra $Cl(V)$ into $\mathcal{A}$. The Clifford algebra $Cl(V)$ is the *unique* associative and multilinear algebra with this property. Thus, for generalizing methods from algebra, analysis, calculus, differential geometry (etc.) of real numbers, complex numbers, and quaternion algebra to vector spaces and multivector spaces (which include additional elements representing all 2D up to $n$D subspaces, i.e., plane elements up to hypervolume elements), the study of Clifford algebras becomes *unavoidable*. Indeed, repeatedly and independently, a long list of Clifford algebras, their subalgebras, and in Clifford algebras, embedded algebras (like *octonions* [8]) of many spaces have been studied and applied historically, often under different names.

Some of these algebras are *complex numbers (and the complex number plane), Wessel algebra, hyperbolic numbers (split complex numbers, real tessarines), dual numbers, quaternions, biquaternions (complex quaternions), dual quaternions, Plücker coordinates, bicomplex numbers (commutative quaternions, tessarines, Segre quaternions), Pauli algebra (space algebra), Dirac algebra (spacetime algebra, Minkowski algebra), algebra of physical space, para-vector algebra, spinor algebra, Lie algebras, Cartan algebra, versor algebra, rotor algebra, motor algebra, Clifford bracket algebra, conformal algebra, projective geometric algebra (PGA), zeon algebra, algebra of differential forms, and so on*; see Hitzer [9].

Regarding the notation of Clifford GAs, a certain variety can be found. A frequently used notation is $Cl(p, q, r)$ for the Clifford GA of a space $\mathbb{R}^{p,q,r}$, with dimension $n = p+q+r$, with an orthonormal basis of $p$ vectors squaring to $+1$, $q$ vector squaring to $-1$, and $r$ vectors squaring to 0. For example, PGA of three dimensions uses $Cl(3, 0, 1)$. Note that for $r = 0$, it is customary to abbreviate $\mathbb{R}^{p,q} = \mathbb{R}^{p,q,0}$ and $Cl(p, q) = Cl(p, q, 0)$. Furthermore, for $q = 0$, many authors abbreviate $\mathbb{R}^n = \mathbb{R}^{n,0}$ and $Cl(n) = Cl(n, 0)$; for example, the GA of three-dimensional Euclidean space $\mathbb{R}^3$ is $Cl(3, 0)$, and CGA for three-dimensional Euclidean space, extended by a Minkowski type plane $\mathbb{R}^{1,1}$ is $Cl(4, 1)$. But note that particularly in the field of Clifford analysis, authors may instead use $\mathbb{R}^n = \mathbb{R}^{0,n}$; it is therefore advisable when reading a publication to first ascertain which notation the author uses. Moreover, one often finds $\mathcal{G}_{p,q,r} = \mathbb{G}_{p,q,r} = G_{p,q,r} = Cl_{p,q,r} = \mathbb{R}_{p,q,r}$, and so on.

The paper is structured as follows, revealing how we clustered and ordered the 101 publications thematically (which is of course somewhat subjective). First, Section 2 refers the reader to similar earlier survey projects conducted during the last 10 years. Then, Section 3 provides an overview of applications of GA for mathematics and computations, while Section 4 introduces applications to higher order surfaces and geometry. Next, Section 5 shows the use of GA for image and signal processing. This is followed by Section 6 on GA computing and GA software and Section 7 on quantum computing with GA. Data processing with GA is featured in Section 8, many applications to neural networks in Section 9, and in Section 10 to the medical field. After that, Section 11 explores applications in physics, that is, mechanics, electrodynamics, gravity, and quantum physics. The new pioneering applications in electric engineering are surveyed in Section 12 and control and robotics in Section 13. The paper is concluded with Section 14 and the list of references.

## 2 | PREVIOUS GA APPLICATION SURVEYS

In Breuils et al. [14], the authors present a survey including applications of Clifford GA in the past decade (mainly within 2013-2021), several of which were presented in the Applied Geometric Algebra for Computer Science and Engineering (AGACSE) conference series, as well as the annual Empowering Novel Geometric Algebra for Graphics and Engineering (ENGAGE) workshops, which are part of the conference Computer Graphics International. Breuils et al. [14] can be seen as a continuation of the earlier survey [15] published in 2013, summarizes approximately 200 GA publications.

It is further supplemented by a similar amount of publications on GA applications in Hitzer et al. [16] of early 2022, surveying publications in the years 2019 to 2022. These surveys include GA applications related to engineering, electric engineering, optical fibers, geographic information systems, geometry, molecular geometry, protein structure, neural networks, artificial intelligence, encryption, physics and software as well as signal, image, and video processing.

[2]The inner product defines the measurement of length and angle.
[3]This setting amounts to an algebra generating relationship.
[4]No product sign will be introduced; simple juxtaposition implies the geometric product just like $2x = 2 \times x$.
[5]A $\mathbb{K}$-isometry between two inner-product spaces is a $\mathbb{K}$-linear mapping preserving the inner products.
[6]A $\mathbb{K}$-algebra is a $\mathbb{K}$-vector space equipped with an associative and multilinear product. An inner-product $\mathbb{K}$-algebra is a $\mathbb{K}$-algebra equipped with an inner product structure when taken as $\mathbb{K}$-vector space.
[7]Important fields are real $\mathbb{R}$ and complex numbers $\mathbb{C}$ and so on.
[8]That is a $\mathbb{K}$-linear homomorphism preserving the inner products; that is, a $\mathbb{K}$-linear mapping preserving both the products of the algebras when taken as rings and the inner products of the algebras when taken as inner-product vector spaces.

## 3 | GA FOR MATHEMATICS AND COMPUTATION

This section is divided into three topical parts: the mathematical structure of GA, application of GA to mathematical problems, and GA-based computing.

The first part consists of papers where the authors study the objects of GA, that is, geometric entities or transformations and functions. Thus, Bayro-Corrochano et al. [17] revisit a proposal for the formulation of objects and geometric relations and constraints in CGA and discuss its application to various engineering disciplines. Representations of transformations, rotors in particular, are then searched for by Lasenby et al. [18] in the form of their reconstruction from initial and final frames. A series of papers by Acus Arturas et al. is handling the closed-form exponentials of multivectors in general dimension [19] with additional conditions on the GA signature, both in low dimensions [20] and generally in Dargys and Acus [21], respectively. In algebras of dimension $n$ less than six, more maps such as normalization, square roots, and exponential and logarithmic maps are treated by De Keninck and Roelfs [22]. As for the operations on GA, Stejskal et al. [23] provide a comprehensive description of relations between objects in 2D space using the matrix product of vectors, the geometric product, and the dot product of complex numbers. Finally, structural issues of the embedding of octonions in GA of all signatures in three and four dimensions are treated by Hitzer [24].

The second part is perfectly represented by a paper of Ramírez et al. [25] where Lyapunov stability theory for smooth nonlinear autonomous dynamical systems is presented in terms of GA or the paper by Abdulkhaev and Shirokov et al. [26] where basis-free formulae for characteristic polynomial coefficients are derived in terms of GA. Structural mathematical concepts, such as the generalization of Lipschitz and spin groups, or twisted algebras of GA are targeted by Filimoshina and Shirokov [27] and Matsuno [28], respectively.

In the last category, the paper by Stephane Breuils et al. [29] discusses the complexity of products in GA, more precisely the number of operations required to compute a product, in a dedicated program for example, and the complexity of enumerating these operations.

## 4 | GA FOR SURFACES AND GEOMETRY

In his new book *Mathematics for Computer Graphics* [30], Vince explains a wide range of mathematical techniques and problem-solving strategies associated with computer games, computer animation, virtual reality, CAD, and other areas of computer graphics (CG). Among the list of worked examples and color illustrations revolving around the mathematics required for CG, the author dedicates a chapter to introduce the basic notation and functionality of GA, as an extension of the complex numbers and quaternions that were described in detail in previous sections. The introduction to GA is performed in the author's characteristic descriptive manner and along with the numerical examples lure the reader to this exciting new world of multivectors.

The GA $\mathbb{G}_{6,3} = Cl(6,3)$, also denoted as *Quadric Geometric Algebra* (QGA), is a known generalization of CGA. Esquivel[31] provides the description of common geometric entities, for example, points, planes, and spheres and also (hyperbolic and parabolic) cylinders, elliptic cones, or ellipsoids. The rotation and revolution of points and quadratic primitives, as well as the vanishing coordinate frame $\{e_{\infty x}, e_{\infty y}, e_{\infty z}\}$, are described in detail, providing better insight of QGA.

As the interpolation of the trajectory of points and geometric entities remains an important problem for kinematics (e.g., movement for robots), several algorithms exist to describe such trajectories, often involving the use of matrices, quaternions, dual quaternions, and the study quadric. Martinez-Terán et al. [32] exploit CGA to represent motors as 8D vectors in projective space $\mathbb{P}^7$, thus reducing the interpolation of rotations and translations to a linear problem. A CPU and a GPU (CUDA) implementation were tested to obtain performance metrics, and the methodology was applied to interpolate trajectories in medical robotics for kidney surgery.

Towards an effective generation of molecular surfaces, Alfarraj and Wei [33] employ Clifford Fourier transforms (CFT), a generalization of the classical FT. Using the CFT in $\mathbb{R}_3 = Cl(3,0)$ allows solving partial differential equations and specifically the ones involved in the mode decomposition process. After setting the theoretical background, authors apply the proposed method to small molecules and proteins, generating their surfaces and comparing their output with other definitions. The importance of their work is further highlighted, as their methodology can be applied for protein electro-static surface potentials and solvation free energy, as well as other biological sciences.

Using quaternion operators on orbits (curves or surfaces) in the Euclidean space $\mathbb{E}^3$ and its GA $Cl(3,0)$, Aslan and Yayli [34] demonstrate how to generate these orbits from points, curves, or surfaces. An explicit form of the motions that achieves this is proved and provided as one- or two-parameter homothetic motions, along with detailed numerical examples.

Identifying sphere intersections is a simply-to-state yet important core problem of many applications, among a variety of scientific fields, for example, data sciences and 3D protein structure determination. When the radii of the spheres are not known, the notion of a *spherical shell* can be employed, by replacing the precise radii with interval values. Considering the intersection of spheres and/or spherical shells in higher dimensions, Lavor et al. [35] provide a methodology to identify and characterize them. A comparison of the theoretical approaches via linear algebra or CGA is provided and illustrated with numerical examples. The result of this comparison further highlights the ability of CGA to naturally preserve the geometric intuition of the problem, even in dimensions higher than three.

To view fractal-based devices not as a problem only related to high-tech innovation but also related as a marketing problem, Mukhopadhyay [36] tracks the development of fractal information theory using a universal geometric musical language and a 12-dimensional GA. It explores how fractal decision-making can be used in the fields of business analytics, security for risk mitigation, and healthcare. See also [37–39].

## 5 | GA FOR IMAGE AND SIGNAL PROCESSING

Hitzer [40] generalizes the spacetime Fourier transform (SFT) of Hitzer [41] to a special affine Fourier transform (SASFT, also known as offset linear canonical transform) for 16-dimensional spacetime multivector $Cl(3,1)$-valued signals over the domain of spacetime (Minkowski space) $\mathbb{R}^{3,1}$. This includes computation in terms of the SFT, its properties of multivector coefficient linearity, shift and modulation, inversion, Rayleigh (Parseval) energy theorem, partial derivative identities, a directional uncertainty principle, and its specialization to coordinates.

Zhang et al. [42] observe that the complex-valued random Fourier GA mapping (CRFGAM) method can solve the overcoupling issue of real and imaginary parts for complex-valued signals. In order to improve the accuracy of nonlinear mapping in the CRFGAM method, they propose a *multidimensional* complex-valued random Fourier GA mapping (MDCRFGAM) method by expanding CRFGAM to a multidimensional mapping space and extend this to a multidimensional complex-valued random Fourier GA least mean square (MDCRFGALMS) algorithm. Related to this, Huang et al. [43] propose a novel *fixed dimensional* adaptive filter for complex-valued signals named complex-valued random Fourier GA least mean square (CRFGALMS). With GA adaptive filtering, real and imaginary parts of complex-valued signals are mapped to random Fourier features space (RFFS) to improve the efficiency of the nonlinear mapping for complex-valued signals. The proposed GA-based mapping has superior presentation abilities in the complex-valued domain and decouples the nonlinear mapping of real and imaginary parts of complex-valued signals. In both cases, simulations on nonlinear channel equalization are conducted for validation.

Xiangyang Wang et al. [44] present two novel GA methods to estimate two-dimensional (2D) directions-of-arrival (DOA) of noncircular (NC) signals for uniform rectangular arrays (URA). Traditional long vector methods inevitably lose orthogonality inside each electromagnetic vector sensor (EMVS) and miss information of second-order statistical properties and increase computational complexity. New GA-based estimating of signal parameters via rotational invariance techniques (ESPRIT) and propagation method (PM) algorithms are proposed. GA maintains the relationship among multidimensional signals. The six EMVS components are represented as a GA multivector, and a GA-based extended covariance matrix utilizes the signal information more completely. The new GA estimation of signal parameters via rotational invariance techniques for NC signal processing (GANC-ESPRIT) yields the DOA with high accuracy. The new GA PM for NC signal estimation (GANC-PM) uses linear transformations to calculate angle parameters. Memory requirement is greatly reduced versus long vector methods. Simulations validate good angular resolution, and complexity analysis shows better performance with reduced computation.

Wang et al. [45] propose two novel GA-based adaptive filtering algorithms based on the minimum error entropy (MEE) criterion and the joint criterion (MSEMEE) of MEE and the mean square error (MSE). Simulation results show that for the mean square deviation (MSD) learning curve, the GA-based MEE (GA-MEE) algorithm has faster convergence rate and better steady-state accuracy compared with the GA-based maximum correntropy criterion algorithm (GA-MCC) under the same generalized signal-to-noise ratio (GSNR). The GA-MEE algorithm reduces the convergence rate but improves the steady-state accuracy by 10–15 dB compared with adaptive filtering algorithms based on GA and second-order statistics. When GA-MSEMEE and the adaptive filtering algorithms based on GA and second-order statistics, respectively, keep the same convergence rate, its steady-state accuracy is improved by 10–15 dB, and when GA-MSEMEE and GA-MEE maintain approximately steady-state accuracy, its convergence rate is improved by nearly 100 iterations. For noise cancelation, the average recovery error of the two new algorithms improves over other GA-based adaptive filtering algorithms. This provides new methods to deal with multichannel interference in wireless networks.

Wang and Wang [46] observe that GA-based adaptive filters have been applied to fields like 3D wind speed, computer vision, and fusion prediction of dynamic pressure. To further improve performance, they propose GA adaptive algorithms convexly combining two different step size GA least mean square algorithms (CGA-LMS) and provide detailed steady-state performance analysis. To address the phenomenon that the slow filter may lag considerably behind the fast filter, which slows down the overall convergence of the combined GA filter, they add a novel instantaneous transfer strategy, creating a CGA-LMS algorithm with transfer strategy (CGA-LMS-TS). To process the NC 3D and 4D signals, they employ a convex combination of a widely linear GA-LMS (CWL-GA-LMS) algorithm with a CWL-GA-LMS with transfer strategy (CWL-GA-LMS-TS). Simulations validate performance and correctness.

Derevianko and Loucka [47] introduce a search for the similarity transformation of two-dimensional point clouds using GA for conics (GAC). They represent image objects by ellipses fitted into contour points. This speeds up consequent similarity search and saves memory. Examples with real object images are included.

Liu et al. [48] propose a GA product expression associated with geometric relationships of vectorized THz refractive index and absorption coefficients. From this expression, candidate characteristic parameters are extracted for liquids discrimination presenting abundant second-order correlation information of optical parameters with rising dimensions. Three groups of liquids, containing C-reactive protein calibrators and alpha fetoprotein calibrators, were used for validation. Comparing with traditional THz parameters of refractive index, absorption coefficient, and complex permittivity, the novel approach is superior in differentiation with the evaluation of statistical differences and effect size.

Calado et al. [49] compare two new geometric model-based approaches to gesture recognition which support the visualization and geometrical interpretation of the recognition process, with two classical ML algorithms, *k*-nearest neighbor (*k*-NN) and support vector machine (SVM), and two state-of-the-art (SotA) deep learning (DL) models, bidirectional long short-term memory (BiLSTM) and gated recurrent unit (GRU), on an experimental Italian Sign Language (LIS) data set. They achieve a compromise between high recognition rates (>90%) and fast recognition times (<0.1 s) adequate for human–computer interaction.

We note that in Section 8, the reader will find information on application to geographic data, GA-based collaborative filtering, enhanced knowledge graph embedding, and secret sharing.

## 6 | GA FOR COMPUTING AND SOFTWARE

Aiming to educate both GA experts and people that are not yet acquainted with the concept of multivectors, Hildenbrand and Rockwood [50] gave a SIGGRAPH course presentation[9] touching various aspects of GA. After a motivating introduction to the work of Grassmann and Clifford, they provided the deep connection between algebra and geometry that holds for the entities of such mathematical frameworks, as well as basic notation and properties. After describing in more detail the two dominant GAs, namely, projective GA (PGA) and CGA, the GAALOP software framework is used to perform various operations. Specifically, GAALOP and its web version, GAALOPWeb, is a well-known software dedicated to optimize GA files. Its usage is demonstrated in a straightforward way for both PGA and CGA, to perform numerical operations as well as to obtain visual results. This work concludes with the presentation of GAC, an algebra that has become increasingly popular due to a diverse range of applications.

By analyzing the majority of approaches used by logicians to support a mathematical claim, one may deduce that there are two ways that a computer can help establish a claim: It can either help find a proof in the first place (*automated theorem proving*) or it can help verify that a purported proof is correct (*interactive theorem proving*).

The *lean theorem prover* aims to bridge the gap between interactive and automated theorem proving, by situating automated tools and methods in a framework that supports user interaction and the construction of fully specified axiomatic proofs. Wieser and Song [51] recently presented the partial formalization of GA in Lean via describing multivectors as the quotient of a tensor algebra by a suitable relation, in a basis-agnostic manner. Although not complete yet, their work[10] is applicable to GA of various dimensions and sets the bases for a complete GA formalization within the promising `mathlib` library, in a future proof and easy-to-extend way.

[9] The complete presentation is available online (https://dl.acm.org/doi/10.1145/3532720.3535655).
[10] This work can be found online (https://github.com/pygae/lean-ga).

In the past decades, quaternions have been the dominant representation form for rotations in the CG pipeline, along with the use of matrices and vectors to store translations and scalings. Although there is concrete evidence that the use of more advanced forms such as dual quaternions and multivectors (for motors, rotors, and dilators in PGA and CGA) can alleviate commonly appearing rendering artifacts, such as the candy-wrapper effect, these forms are usually not incorporated in the modern CG curriculum, mainly due to the lack of support from the frameworks used. Papagiannakis et al. [52] present the *Elements* project, a pythonic framework that is based on the entity-component-system principle on top of a scene graph-based implementation. Using Elements, users may create and render a 3D scene, using transformation data that are either of the traditional form, that is, a TRS matrix, or any other potential form, such as quaternions, dual quaternions, or multivectors. See also [53, 54]

## 7 | GA FOR QUANTUM COMPUTING

Hrdina et al. [55] investigate the representation of $n$-qubits and quantum gates acting on them as elements of a complex GA defined on a complex vector space of dimension $2n$. In such spaces, the Dirac formalism can be realized straightforwardly. Aiming to establish GA as a major language for quantum computing, the authors introduce the noncomplex Quantum Register Algebra (QRA) [56] and exploit the GAALOP (Geometric Algebra Algorithms Optimizer) framework to perform numerical operations. Their latter work proves advantageous for quantum computing beginners since they only have to know GA in order to intuitively describe the objects and operations of quantum computing.

In the same context, Soiguine [57] demonstrates how the the double split experiment results can be resolved with diffraction patterns inherent to wave diffraction. In his work, he exploits the GA formalism along with generalization of complex numbers and subsequent lift of the two-dimensional Hilbert space-valued qubits to geometrically feasible elements of an even GA subalgebra. See also [58].

## 8 | GA IN DATA PROCESSING

Addressing the demand for multilevel declassification of geographic vector field data (GVFD), Luo et al. [59] employ GA to uniformly express it as a GA object and then sequentially apply a rotor and a perturbation operator. Declassifying the final output and comparing it to the input shows that despite the alteration of the original vector field, some general geo-spatial features are retained. This approach offers effective multilevel controls and has good randomness; it suggests a viable solution for data disclosure, secure transmission, and encapsulation storage of such types of data.

Jiang et al. [60] conducted an analysis of the spatial geometric similarity computation based on CGA, aiming to shed light on spatial analysis and data retrieval. By (a) developing a unified expression model for spatial geometric scenes, (b) integrating shapes of objects and spatial relations between them, and (c) establishing a model for spatial geometric similarity computation, they are able to derive measurement information and topological relations of spatial objects. Their method involves simple inner, outer, and geometric product operations and can be applied to retrieve spatial scenes with objects of different types, yielding satisfying results.

As the world is evolving from a binary space to a ternary space, a suitable framework to capture diverse geographic information is required. The seven-dimensional framework proposed in Lv et al. [61] exploits a form of GA representation that is suitable to represent characteristics such as semantics, attribute, interrelationship, and interplay mechanisms, which are usually unstructured and cannot be represented as plain geometries or simple algebraic equations.

Wu et al. [62] propose a GA-based collaborative filtering model for recommendation, based on representation learning. Their methodology involves the usage of multivectors to represent users and items, while plain geometric and inner product operations are used to indicate the historical interaction between users and items. Training the model with the resulting multivectors, interdependencies between components of multivectors can be predicted, allowing more complex interactions between users and items to be captured.

A quite intriguing area where GA proves to yield enhanced results involves KGE, which introduces important challenges for knowledge representation learning such as the management of time-evolving data nodes. Link prediction tasks in KGE have shown promising performance, especially due to complex or hypercomplex representation forms. Xu et al. [63] present a GA-based approach that achieves or even surpasses the state-of-the-art performance threshold.

Their approach involves data representation as multivectors of $\mathbb{G}^2 = Cl(2,0)$ or $\mathbb{G}^3 = Cl(3,0)$, as well as a GA-based model that consumes the obtained data set. This novel, yet effective, approach for KG, manages to successfully handle the temporal KGE completion problem, by exploiting the geometric meaning of the time embedding.

A variant of the well-known threshold secret sharing scheme proposed by Adi Shamir in 1979 is presented in Silva et al. [64]. Its methodology involves using multivector objects to be secrets in a secret sharing scenario without incurring any additional overhead in comparison with the reference scheme. Fundamental computations such as addition and multiplication can be performed over random shares, making this approach ideal to use the context of multisecret sharing. As both secret and random shares of members are members of the same space, the fundamental property of Shamir's scene, *idealness*, is also preserved.

We note that in Section 5, the reader will find a discussion of GA applications to integral transforms of spacetime signals, CRFGAM, direction of arrival problems, GA-based adaptive filtering, point cloud similarity, vectorized THz data, and gesture recognition. See also [65].

## 9 | GA APPLIED TO NEURAL NETWORKS

GA seems to be well-suited regarding the representation of multidimensional data. Expressing such data in multivector form, Li et al. [66] were able to capture the inherent structures and preserve the correlation of multiple dimensions in the context of a *long and short-term time series* network that processes multivariate time series. Their method outperforms traditional techniques with higher prediction accuracy.

Following a similar approach, Li et al. [67] propose a GA-based mapping of each spectrum in a hyperspectral image (HSI); the multivectors derived are then used in a convolutional neural network (CNN) that deals with multichannel HSIs. In such an approach, increased performance, less overfitting risks, and better information preservation were achieved with respect to identical real-valued CNNs.

As graph neural networks become increasingly popular, the authors of Liu and Cao [68] proved that graph feature embedding in GA can improve the quality of graph feature presentation. Using a *few shot cross-domain* classification task as application, their proposed approach yields improved results over metric-based methods as it uses the high algebraic dimensions of GA to reduce the distortion of feature information despite the increased hidden layers.

The paper [69] establishes a parametrization of hypercomplex convolutional layers and introduces a family of parametrized hypercomplex neural networks (PHNNs) that are lightweight and efficient large-scale models. Convolution rules and the filter organization are directly inferred from the data without requiring a rigidly predefined domain structure. PHNNs operate flexibly in any user-defined or tuned domain, from one to $n$ dimensions, regardless of whether the algebra rules are preset. This allows processing multidimensional inputs in their natural domain without annexing further dimensions, as done, instead, in quaternion neural networks (QNNs) for three-dimensional inputs like color images. As a result, the proposed family of PHNNs operates with $1/n$ free parameters compared with the real domain. For image and audio datasets better performance than real and quaternion-valued counterparts is shown.

Robust and efficient transmission of data over networks is crucial for collaborative AR/VR applications. In such a context, the authors of Kamarianakis et al. [70] compare traditional representation forms (vectors and matrices) with GA-based forms (quaternions, dual quaternions, and PGA/CGA multivectors) for transmission, recording, and replay purposes. Their work suggests that, regarding transmission, GA is the only viable solution for poor network conditions and yields better or on par results when no such conditions occur. It is also shown that, using GA forms, less transformation data per second can be recorded without impacting the interpolated keyframes during replay.

Regarding computer-vision oriented deep learning techniques, GA is a great means to capture rotational data that constantly appear, as many of the related tasks such as pose estimation from images or point clouds can be formulated as a regression on rotations. Overcoming limitations posed by commonly used representations, Pepe et al. [71] exploit multivectors to reduce errors in high-noise datasets, while learning fewer parameters.

The work [72] introduces Pin $(p, q, r)$ group action layers, which linearly combine object transformations. Together with a new activation and normalization scheme, these layers serve as adjustable geometric templates that can be refined via gradient descent. Theoretical advantages are strongly reflected in the modeling of three-dimensional rigid body transformations as well as large-scale fluid dynamics simulations, with significantly improved performance.

With the development of cities and the increased demand for traffic management, predicting traffic data have become an increasingly researched yet complicated task due to changeable and complex traffic conditions.

By capturing related traffic information as multivectors, the multidimensionality of the data can be maintained and complex features can also be extracted in the context of spatio-temporal attention neural networks [73], multichannel residual networks [74], generative adversarial networks [75], or graph attention networks [76]. These works suggest an improvement of performance when GA-based forms are used compared with real data-based implementations and are suitable to predict even long-term features such as the traffic speed for a whole day.

Alongside algorithmic advances in neural networks, new technological paradigms are developed where machine and deep learning techniques are incorporated in dedicated hardware. Such an example is described in García-Limón et al. [77], where a hypersphere neural network for energy consumption monitoring, based on GA representation of points and hyperspheres, is implemented in an IoT device, consisting of a NodeMCU board and an Esp8266 microcontroller.

Towards solving the problem that derives from nonorthogonal data attributes in conventional machine learning applications, Oktar and Turkan [78] propose a shift-invariant $k$-means methodology. The authors suggest the use of complex, hypercomplex, and GA-based approaches and encoding schemes, to better capture invariance under rotation or general transformations, via convolution operators that preserve spatio-temporal information.

GA-based neural networks also find applications in the medical field.

## 10 | GA IN MEDICAL SCIENCE

GA is able to positively impact the majority of scientific areas, including the medical sciences. A good example is provided in Tu et al. [79] where the most common neurodegenerative disorder, the Alzheimer's disease (AD), is considered. Introducing a GA-based multimodal feature transformation and fusion model, a fast-convergent artificial neural network framework can provide a highly accurate AD diagnosis.

Further proving that high-dimensional information can be more efficiently handled via GA, Wang et al. [80] propose a multimodal medical image fusion algorithm based on a discrete GA cosine transform. In their work, they conduct fusion experiments on four groups of brain medical color images, by considering the connection between the color image channels and using multivectors to represent the source image. Results indicate improved performance, but only marginal advantage compared with traditional algorithms, thus leaving room for future improvements.

A similar technique is exploited in Li et al. [81] where GA is employed for (a) a multimodal medical image fusion algorithm, (b) an orthogonal matching pursuit algorithm, and (c) a $K$-means clustering singular value decomposition algorithm. All these components are involved into an effective algorithm that avoids losing the correlation of color channels of medical images and surpasses the state-of-the-art approaches in terms of subjective and objective quality evaluation.

GA is also employed in the context of modern applications that impact the medical training landscape. Zikas et al. [82] propose MAGES 4.0, a novel Software Development Kit (SDK), to accelerate the creation of collaborative medical training applications in VR/AR. Their solution is essentially a low-code metaverse authoring platform for developers to rapidly prototype high-fidelity and high-complexity medical simulations. Among the variety of novelties it incorporates, this Unity3D-based framework exploits GA-based representation forms, namely, dual quaternions and multivectors, to efficiently transmit user actions over the network in multi-user collaborative scenarios. Exploiting the mechanisms described in Kamarianakis et al. [70], its under-the-hood GA interpolation engine achieves optimal performance compared with state-of-the-art. The same engine and all-in-one GA framework is also employed to effectively record and replay VR/AR sessions.

See also the discussion of Alfarraj and Wei [33] and Martinez-Terán [32] in Section 4. See also [83].

## 11 | GA IN PHYSICS

### 11.1 | Mechanics

Jensen and Poling [84] describe angular momentum with $Cl(3,0)$ bivectors, visualized as *tiles* with area and orientation whose components form an antisymmetric matrix. Bivectors have historically been considered mostly in specialized contexts like spacetime classification or GA but are no more complicated than cross products. Teaching rotational physics in this language is ultimately viewed more fundamental and helps to understand rotations in relativity and extra dimensions.

According to Brechet [85] in $Cl(3,0)$, the Poisson formula for the time derivative of unit vectors of a moving frame is expressed by the angular velocity bivector and applied to cylindrical and spherical frames. The rotational dynamics of a point particle and a rigid body are fully determined by the time evolution of $Cl(3,0)$ rotors. The mapping of the angular velocity bivector onto the angular momentum bivector is the inertia map. It is characterized by symmetric coefficients (moments of inertia) in the (rigid body) principal axis frame. The Huygens–Steiner theorem, the kinetic energy of a rigid body and the Euler equations are expressed in terms of bivector components. The rotational dynamics of a gyroscope provides an example.

Ikemori et al. [86] formulate the Runge–Lenz vector in the Kepler problem as a three-dimensional GA projection of a $SO(4)$ moment map that acts on the phase space of a four-dimensional particle motion. The Runge–Lenz vector originates from geometric symmetry of $\mathbb{R}^4 \times \mathbb{R}^4$ phase space.

## 11.2 | Electrodynamics

Brechet [87] treats electrodynamics of electric charges and currents in vacuum and dielectric and magnetic material media, using $Cl(3,0)$ and STA $Cl(1,3)$. With a polarization multivector and an auxiliary electromagnetic field multivector, Maxwell's equation is formulated in a material medium in $Cl(3,0)$ and in STA with an extra bound current vector. The wave equation in a material medium is obtained from the gradient of the Maxwell equation. For a uniform electromagnetic medium of induced electric and magnetic dipoles, the stress-energy momentum vector is formulated with the electromagnetic force density vector as inhomogeneity, and the Maxwell equation in a material medium is written in STA as vector potential wave equation.

According to Sen [88], STA provides an invariant description of electromagnetic theory, without reference to any inertial system. Using elementary geometric calculus, STA allows the direct analytical introduction of magnetic monopoles and renders the equations for both constituent fields, symmetric, and inhomogeneous. STA unifies the Lorentz force equation and the electromagnetic power equation.

## 11.3 | Gravity

Noticing that the even subalgebras of $Cl(3,1)$ and $Cl(1,3)$ can algebraically not be distinguished, Wu [89] provides a signature invariant treatment of general Lorentz boosts and general spatial rotations in arbitrary planes. For a massive particle, the spacetime splits of the velocity, acceleration, momentum, and force four-vectors with the normalized four-velocity of the fiducial observer, at rest in the coordinate system of the spacetime metric, are given, where the proper time of the fiducial observer is identified, and the contribution of the bivector connection is considered, and with these results, a three-dimensional analog of Newton's second law for this particle in curved spacetime is achieved. As example, in Lense–Thirring spacetime, the signature invariant precessional angular velocity of a gyroscopic spin is derived in STA.

Xu [90] discusses in STA gravitational wave solutions of the gauge theory of gravity field equations with a negative cosmological constant and shows that these solutions are of Petrov type-$N$. He also discusses the velocity memory effect by calculating the velocity change of an initially free falling massive particle due to the presence of gravitational waves.

## 11.4 | Quantum physics

Because geometric algebra of spacetime (STA) is isomorphic to the $4 \times 4$ complex Dirac matrix algebra of Dirac's equation for relativistic electrons, applications of geometric algebra to quantum mechanics have a long and rich tradition.

Hamilton [91] shows how it is possible to start with row and column spinors and their products which include GA multivectors to construct a *super GA* for both spinors and GA, thus viewing fermions (spinors) as truly fundamental and expressing the exclusion principle as elementary spinor product rule. Lasenby [92] embeds octonions[11] in STA such that the octonion product norm corresponds to the preservation of the timelike part of a particle Dirac current. This allows to embed and geometrically interpret earlier work of particle physics based on octonions. Furthermore, $SU(3)$ of six Euclidean dimensions can also be embedded in STA based on bivector norm preservation. Finally, interesting connections to $G_2$ and $SU(8)$ are considered.

Zatloukal [93] examines the minimal coupling procedure in Hestenes' STA Dirac equation, where spinors are identified with even multivectors, and finds a non-Abelian generalization of the electromagnetic gauge potential. Marks [94] observes that real GAs $Cl(n;s) = Cl(p,q)$, $s = p-q$, are periodic in $s$, that is, $Cl(n;s) \cong Cl(n;s+8k)$ for $p+4k \geq 0, p-4k \geq 0$,

[11]See the further expansion of this embedding of octonions in other GAs in Hitzer [24] discussed on page 6.

and recursive in $n$, that is, $Cl(n;s) \otimes Cl(2;0) = Cl(n+2;s)$. Moreover, in GA, two-dimensional Euclidean planes and spacetime Minkowski planes have isomorphic GAs. Their direct product gives STA. A further product algebraically describes strong forces and then generates a lattice with standard model physics at each node. This implies Noether's theorem conservation laws from uniformity of spacetime based on the recursive generation.

Soiguine [95] begins by replacing complex numbers by real GA multivectors, achieving clear explanations of atoms as a kind of planetary system. The three-sphere $S^3$ hosts torsion-like states eliminating abstract Hilbert space vectors. $S^3$ points evolve, governed by the updated Schrödinger equation, and act in measurements on observables as operators.

In Soiguine [96], he furthermore explains double split experiments in GA, where particles create diffraction patterns inherent to wave diffraction, eliminating a key difficulty in the interpretation of quantum mechanics.

Andoni [97] uses the spin-position decoupling approximation, to substitute the Pauli vector-matrix spin model by a vector with a phase in 3D orientation space endowed with GA. He explains the resulting properties, including measurement irreversibility, entanglement, and 2D in 3D spin space embedding. The formalism appears in two complementary "spinor" or "vector" forms, providing a clear geometric picture of spin correlations and transformations entirely in the 3D physical orientation space.

Trell [98] examines Marius Sophus Lie's 1871 PhD thesis *Over en Classe geometriske Transformationer*, relates it to Hermann Grassmann's *Ausdehnungslehre* (1844/1862) and William Kingdom Clifford's *Space-theory of Matter* (1876), and develops a concrete cellular automaton building kit of the standard model organized as structural $\mathbb{R}^3 \times SO(3)$ wave-packets, both inwards from the elementary particles and outwards via the periodic table of the atoms over further hierarchical growth in molecular and crystal stages to an isotropic space-filling of the whole classical Euclidean Universe in harmonic exchange with its relativistic spherical moiety, dark mass, and energy.

Christian [99] tries to further explain his model for local origins of quantum correlations, assumed to be rooted in GA, and to defend it against some criticism. See also [100].

## 12 | GA FOR ELECTRIC ENGINEERING

Eid and Montoya [101] explore the concept of generalized geometrical frequency in electrical systems with an arbitrary number of phases by using GA and differential geometry. With the *Darboux bivector* concept, they can find a bivector that encodes the invariant geometrical properties of a spatial curve named electrical curve. The traditional concept of instantaneous frequency in power networks can be intimately linked to the Darboux bivector. Several examples illustrate this result.

Eid and Montoya [102] then present a new GA framework for a systematic generalization of well-known instantaneous transformations used in electrical engineering for power system analysis and computing through geometric principles. By introducing the Kirchhoff vector and Kirchhoff subspace, a new generalized transformation is presented that unifies the Clarke, Park, or hyperspace vector transformations (widely used in electrical engineering) as particular cases. A generalization to an arbitrary number of phases is achieved. All the underlying ideas are presented by means of space-like conceptualizations, substantiated by their corresponding algebraic formulation. This proposal has wide ranging power system applications such as to electrical machines, current compensation, power quality, electronic converters, or transmission lines. Preliminary results show superior efficiency compared with matrix methods. Some real-world examples are included.

Montoya et al. [103] establish an alternative physical formulation for the harmonic power flow in electrical systems provided by GA, the Poynting vector (PV), and the Poynting theorem (PT). Given the traditional definition of PV (Abraham approach) as cross product of electric and magnetic field vectors, they exploit the duality of the cross product to the much more powerful wedge product of exterior algebra. Using vector space concepts, they develop a completely GA-based approach founded on top of the isomorphism among periodic time-domain signals and Euclidean spaces. This sheds light on the long-running discussion of electric power flow in nonsinusoidal and nonlinear electrical power systems.

Then Montoya et al. [104] for the first time model power flows in electrical circuits in a mixed time-frequency domain by using GA and the Hilbert transform. They overcome some of the limitations of the existing methodologies, in which the so-called *active current* may not lead to the lowest root mean square (RMS) current under distorted supply or unbalanced load. Moreover, this current may contain higher levels of harmonic distortion compared with the supply voltage. The proposed method can be used for sinusoidal and nonsinusoidal power supplies, nonlinear loads, and single-phase and multiphase electrical circuits, and it provides meaningful engineering results with a compact formulation. It can also serve as an advanced tool for developing algorithms in the power electronics field. Several examples verify the approach.

The circuit analysis approach based on GA and $\boldsymbol{M}$ and the power definition based on the geometric product between the voltage and the current multivectors are used by Castro-Núñez et al. [105] to demonstrate shortcomings of the traditional definition of the nonsinusoidal apparent power $S$. The shortcomings of $S$ are illustrated as follows: firstly, by showing an example of how the norm of $\boldsymbol{M}$ contains $S$; secondly, through six experiments that involve compliance with: Kirchhoff's circuit laws, Tellegen's theorem, the principle of conservation of energy, the equivalency of two terminal networks, and the concept of reactive power compensation; and lastly, by showing how the use of $S$ leads the current's physical component power theory astray. The experiments show contradictions between the aforementioned circuit theory fundamentals and the results attained with $S$ but a compelling harmony with the results attained with $\boldsymbol{M}$. The evidence reveals (1) that mathematical models aimed at explaining energy flow in nonsinusoidal circuits should not be based on the traditional decomposition of $S$ and (2) the inappropriateness of extrapolating definitions from sinusoidal to nonsinusoidal settings.

Finally, Sundriyal et al. [106] show that for linear and nonlinear nonsinusoidal circuit conditions, a consensus can be reached on norms that comply with well-known, established standards. They compare the use of the harmonic domain and GA in circuits with disturbances for sinusoidal and nonsinusoidal excitations in order to demonstrate the accuracy of GA in power flow calculations.

## 13 | GA FOR CONTROL AND ROBOTICS

In previous years, GA has been used mostly to unify various frameworks of screw theory, Lie algebra, and dual quaternions. Even now, papers on specific aspects of motion descriptions may be found; see, for example, Ben Cross et al. [107]. The simplicity of motion, that is, Euclidean transformations and their interpretation have been heavily used in robot kinematics. Indeed, papers providing description of robotic motion planning are, for example, Jesús Medrano-Hermosilo et al. [108] for 6-DOF serial robot's forward kinematics or Edgar Macias-Garcia et al. [109] solving the inverse position kinematics for $n$-degrees-of-freedom kinematic chains with revolving joints, and similarly, Lechuga-Gutierrez et al. [110] present a set of generalized iterative algorithms for these mechanisms. A planar three-revolute (3R) serial chain motion generation by establishing the relative kinematics model based on CGA is introduced in Lei Wang et al. [111]. Finally, a new coordinate-invariant geometric constraint equation for 3-RPR planar parallel mechanisms is elaborated by Zhu et al. [112].

Recently, specific GA algorithms focusing on various robotic mechanisms have been developed. Thus, for instance, Li et al. [113] propose a novel 6-DOF platform for pose adjustment of heavy equipment. Also, kinematics of various types of serial robots have been designed in GA; see Tong et al. [114] for a study of spherical 2R mechanism in order to design a seedling pick-up mechanism or Xu [115] for a design of a new parallel manipulator with two rotational and one translational motion. Dynamics of novel robotic mechanisms is also solved in GA; see, for example, Song et al. [116] for dynamic modeling and generalized force analysis of a three-(rotation pair)-(prismatic pair)-spherical pair (3-RPS) parallel mechanism.

In combination with questions of kinematics, associated problems are solved in GA too. For instance, singularities of serial robots are tackled by Zaplana et al. [117], and binocular visual control for a 6-DOF robotic manipulator is described by Stodola et al. [118]. Apart from the control for specific robots or associated problems, there is a group of papers that generalize the concept of kinematic chains in terms of GA. Thus, Zaplana et al. search for closed-form solutions for the inverse kinematics of serial robots [119], and Kalkan et al. [120] introduce the study variety of conformal (CGA) kinematics, that is, a generalization of the study quadric model of rigid body kinematics.

To spread the information about GA, authors integrate new methods and models with standard approaches. Thus, the connection between GA and control on Lie groups is discussed by Hrdina et al. [121], and Bayro-Corrochano et al. [122] show the importance of the Hamiltonian in control theory in terms of GA.

A relatively new type of kinematics studied by means of GA is one of machining with three or five axis. Although the algorithms are not too complicated, the spread of GA to this area is interesting. Chen et al. [123] propose an approach that can generate a smooth tool path that passes through the discrete cutter locations given in the original linear segments analytically and discuss the impact on accuracy. Another approach to volumetric accuracy of a five-axis machine tool can be found in Navrátilová et al. [124].

Apart from controlling algorithms, hardware and computational aspects are also treated. Thus, Uzunović et al. [125] present a combination of two methods that can be effectively combined for control of electrical machines. For computational acceleration, Vitabile et al. [126] propose a novel embedded coprocessor for accelerating CGA operations in robotic tasks. Furthermore, Ramírez et al. [127] present a general method, based on GA, for the synthesis of sliding mode controllers in SISO switched nonlinear systems, and the same authors in an earlier study [128] examine the sliding mode existence conditions, the switching policy, the invariance conditions, the associated equivalent control, and the characterization of ideal sliding dynamics of sliding mode controllers from a GA viewpoint.

Derevianko and Vašík [129] exploit GAC ability to represent transformed objects such as rotated ellipses to propose a novel algorithm for finding the optimal control of a switched dynamical systems with purely imaginary eigenvalues. Such a geometric approach guarantees the optimality of the switching path and eliminates the need for any solver, thus yielding results with minimal numerical errors. See also [130].

## 14 | CONCLUSION

We hope to provide with our survey a comprehensive insight for the reader on new applications of Clifford's geometric algebras published during the last 12 months, prior to writing it (mid-January 2023). The use of the search engine Dimensions.ai ensures a relatively complete overview of the relevant literature, where apart from scientific journal articles, preprints, book chapters (proceedings) and books are also included. Based on this survey, one can expect that GA will continue to be heavily used in physics and find more and more applications in engineering (namely electric engineering), in neural networks and artificial intelligence, and in image and signal processing.

### AUTHOR CONTRIBUTIONS

E. H. did the Dimensions.ai search, clustering, wrote abstract, introduction, conclusion and the sections on Physics, Image and Signal Processing, and Electric Engineering. P. V. wrote the sections on Mathematics and Computing and on Control and Robotics. M. K. wrote all other sections under the guidance and supervision of G. P.

### ACKNOWLEDGEMENTS

E. H. wishes to thank God: *Then God saw everything that He had made, and indeed it was very good.* (Genesis 1:31, NKJV) and his dear family. He thanks his university colleagues for taking the best care of all administrative matters. All of us thank the organizers of *Computer Graphics International* for hosting the *Empowering Novel Geometric Algebra for Graphics and Engineering* (ENGAGE 2022) workshop.

### CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interests.

### ORCID

*Eckhard Hitzer* https://orcid.org/0000-0002-5587-6750
*Manos Kamarianakis* https://orcid.org/0000-0001-6577-0354
*George Papagiannakis* https://orcid.org/0000-0002-2977-9850
*Petr Vašík* https://orcid.org/0000-0001-6725-7378

### REFERENCES

1. W. K. Clifford, *Applications of Grassmann's extensive algebra*, Am. J. Math. **1** (1878), no. 4, 350–358. https://www.jstor.org/stable/2369379
2. H. G. Grassmann, *Extension theory (Die Ausdehnungslehre von 1862)*, Vol. **19**, History of Mathematics, Sources, American Mathematical Society, Rhode Island, London Mathematical Society, 2000. L.C. Kannenberg (translator).
3. W. R. Hamilton, *Elements of quaternions*, Cambridge University Press, Cambridge (UK), 2015. Edited by William Edwin Hamilton, firs published 1866, DOI 10.1017/CBO9780511707162.
4. D. Hestenes, H. Li, and A. Rockwood, *New algebraic tools for classical geometry*, Springer, Berlin, 2001, pp. 3–27. in Sommer G. (ed.), Geometric Computing with Clifford Algebras, DOI 10.1007/978-3-662-04621-0_1.
5. L. Dorst, D. Fontijne, and S. Mann, *Geometric algebra for computer science: an object-oriented approach to geometry*, Morgan Kaufmann Series, Elsevier, Burlington (MA), 2007.
6. E. Hitzer, K. Tachibana, S. Buchholz, and I. Yu, *Carrier method for the general evaluation and control of pose, molecular conformation, tracking, and the like*, Adv. App. Cliff. Alg. **19** (2009), no. 2, 339–364, DOI 10.1007/s00006-009-0160-9.
7. D. Hestenes and G. Sobczyk, *Clifford algebra to geometric CalculA unified language for mathematics and physics*, Kluwer, Dordrecht, 1992. reprinted with corrections.

8. P. Lounesto, *Clifford algebras and spinors*, Cambridge University Press, Cambridge (UK), 2001.
9. E. Hitzer, *Introduction to Clifford's geometric algebra*, SICE J. Control, Measure. Syst. Integr. **51** (2012), no. 4, 338–350. Preprint: https://arxiv.org/abs/1306.1660
10. M. I. Falcao and H. R. Malonek, *Generalized exponentials through Appell sets in* $\mathbb{R}^{n+1}$ *and Bessel functions*, AIP Conf. Proc. **936** (2007), 738–741, DOI 10.1063/1.2790257.
11. S. Breuils, K. Tachibana, and E. Hitzer, *Introduction to Clifford's geometric algebra*, Vixra.org (2021), 1–10. Preprint: https://vixra.org/abs/2108.0145
12. H. Li, *Invariant algebras and geometric reasoning*, WORLD SCIENTIFIC, 2008. https://www.worldscientific.com/doi/abs/10.1142/6514
13. I. R. Porteous, *Clifford algebras and the classical groups*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1995.
14. S. Breuils, K. Tachibana, and E. Hitzer, *New applications of Clifford's geometric algebra*, Adv. Appl. Clifford Algeb. **32** (2022), no. 2, 17.
15. E. Hitzer, T. Nitta, and Y. Kuroe, *Applications of Clifford's geometric algebra*, Adv. Appl. Clifford Algeb. **23** (2013), no. 2, 377–404.
16. E. Hitzer, C. Lavor, and D. Hildenbrand, *Current survey of Clifford geometric algebra applications*, Math. Methods Appl. Sci. (2022), DOI 10.1002/mma.8316.
17. E. J. Bayro-Corrochano, G. Altamirano-Escobedo, A. Ortiz-Gonzalez, V. Farias-Moreno, and N. Chel-Puc, *Computing in the conformal space objects, incidence relations, and geometric constrains for applications in AI, GIS, graphics, robotics, and human-machine interaction*, IEEE Access **10** (2022), 112742–112756. https://ieeexplore.ieee.org/ielx7/6287639/9668973/09926065.pdf
18. A. Lasenby, J. Lasenby, and C. Matsantonis, *Reconstructing a rotor from initial and final frames using characteristic multivectors: with applications in orthogonal transformations*, Math. Methods Appl. Sci. (2022), DOI 10.1002/mma.8811.
19. A. Acus and A. Dargys, *Coordinate-free exponentials of general multivector in* $Cl(p,q)$ *algebras for* $p+q=3$, Math. Methods Appl. Sci. (2022), DOI 10.22541/au.163384987.72001728/v1.
20. A. Dargys and A. Acus, *Exponential and logarithm of multivector in low-dimensional* $(n=p+q<3)$ *Clifford algebras*, Nonlin. Anal. Modell. Control **27** (2022), no. 6, 1129–1149. https://www.journals.vu.lt/nonlinear-analysis/article/download/29528/28672
21. A. Dargys and A. Acus, *Exponentials of general multivector in 3D Clifford algebras*, Nonlin. Anal. Modell. Control **27** (2022), no. 1, 179–197. https://www.journals.vu.lt/nonlinear-analysis/article/download/24476/24634
22. S. De Keninck and M. Roelfs, *Normalization, square roots, and the exponential and logarithmic maps in geometric algebras of less than 6D*, Math. Methods Appl. Sci. (2022). https://arxiv.org/pdf/2206.07496
23. T. Stejskal, J. Svetlík, and M. Lascsáková, *Tensor of order two and geometric properties of 2D metric space*, Mathematics **10** (2022), no. 19, 3524. https://www.mdpi.com/2227-7390/10/19/3524/pdf?version=1665475168
24. E. Hitzer, *Extending Lasenby's embedding of octonions in space-time algebra* $Cl(1,3)$*, to all three- and four dimensional Clifford geometric algebras* $Cl(p,q)$*,* $n=p+q=3,4$, Math. Methods Appl. Sci. (2022), DOI 10.1002/mma.8577.
25. H. S. Ramírez, B. C. Gómez-León, and M. A. Aguilar-Orduña, *Lyapunov stability: a geometric algebra approach*, Adv. Appl. Clifford Algeb. **32** (2022), no. 2, 26. https://app.dimensions.ai/details/publication/pub.1146414986
26. K. Abdulkhaev and D. Shirokov, *Basis-free formulas for characteristic polynomial coefficients in geometric algebras*, Adv. Appl. Clifford Algeb. **32** (2022), no. 5, 57. https://arxiv.org/pdf/2205.13449
27. E. Filimoshina and D. Shirokov, *On generalization of Lipschitz groups and spin groups*, Math. Methods Appl. Sci. (2022). https://arxiv.org/pdf/2205.06045
28. M. Matsuno, *Twisted algebras of geometric algebras*, Canad. Math. Bull. (2022), 1–16. https://arxiv.org/abs/2205.00723
29. S. Breuils, V. Nozick, and A. Sugimoto, *Computational aspects of geometric algebra products of two homogeneous multivectors*, Adv. Appl. Clifford Algeb. **33** (2022), no. 1, 4. https://arxiv.org/pdf/2002.11313
30. J. Vince, *Mathematics for computer graphics*, Springer, 2022.
31. J. C. Z. Esquivel, *Vanishing vector rotation in quadric geometric algebra*, Adv. Appl. Clifford Algeb. **32** (2022), no. 4, 46.
32. G. Martinez-Terán, O. Ureña-Ponce, G. Soria-García, S. Ortega-Cisneros, and E. Bayro-Corrochano, *Fast study quadric interpolation in the conformal geometric algebra framework*, Electronics **11** (2022), no. 10, 1527.
33. A. Alfarraj and G.-W. Wei, *Geometric algebra generation of molecular surfaces*, J. R. Soc. Interf. **19** (2022), no. 189, 20220117.
34. S. Aslan and Y. Yayli, *Motions on curves and surfaces using geometric algebra*, Commun. Facult. Sci. Univ. Ankara Ser. A1 Math. Stat. **71** (2022), no. 1, 39–50.
35. C. Lavor, R. Alves, and L. A. F. Fernandes, *Linear and geometric algebra approaches for sphere and spherical shell intersections in* $\mathbb{R}^n$, Expert Syst. Appl. **187** (2022), 115993.
36. M. Mukhopadhyay, *Going beyond Turing-a fractal approach towards decision making. October 30, 2022. Blackboard, Bulletin of the Mathematics Teachers' Association (India), Issue 5. Forthcoming 2023.* Available at SSRN: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4262323 or DOI 10.2139/ssrn.4262323.
37. K. O. Klausen, *Visualizing stokes theorem with geometric algebra*, 2022. arXiv, DOI 10.48550/arxiv.2206.07177.
38. J. Jost and W. Wenzel, *Geometric algebra for sets with betweenness relations*, Beiträge zur Algebra und Geometrie / Contrib. Algeb. Geomet. **64** (2023), 555–579, DOI 10.1007/s13366-022-00648-w.
39. A. L. G. Mandolesi, *Compendium on multivector contractions*, 2022. arXiv, DOI 10.48550/arxiv.2205.07608.
40. E. Hitzer, *Special affine Fourier transform for space-time algebra signals in detail*, Adv. Appl. Clifford Algeb. **32** (2022), no. 5, 60. https://app.dimensions.ai/details/publication/pub.1151632056
41. E. Hitzer, *Quaternion Fourier transform on quaternion fields and generalizations*, Adv. Appl. Clifford Algeb. **17** (2007), no. 3, 497–517, DOI 10.1007/s00006-007-0037-8.
42. Z. Zhang, P. Cai, M. Shen, G. Huang, and S. Wang, *Geometric algebra adaptive filter based on multi-dimensional complex-valued random Fourier features*, 2022 IEEE 32nd International Workshop on Machine Learning for Signal Processing (MLSP), 2022, pp. 1–6. https://ieeexplore.ieee.org/document/9943322

43. G. Huang, M. Shen, D. Lin, L. Qi, J. Qian, and S. Wang, *Complex-valued random Fourier geometric algebra adaptive filtering*, IEEE Trans. Circ. Syst. II Express Briefs **69** (2022), no. 4, 2346–2350. https://app.dimensions.ai/details/publication/pub.1144591833
44. X. Wang, X. Lv, and R. Wang, *Geometric algebra based 2D-DOA estimation for non-circular signals with an electromagnetic vector array*, Res. Square (2022). https://www.researchsquare.com/article/rs-1619433/latest.pdf
45. R. Wang, Y. Wang, Y. Li, and W. Cao, *Research on geometric algebra-based robust adaptive filtering algorithms in wireless communication systems*, EURASIP J. Wirel. Commun. Netw. **2022** (2022), no. 1, 38, DOI 10.1186/s13638-022-02100-y.
46. W. Wang and J. Wang, *Convex combination of two geometric-algebra least mean square algorithms and its performance analysis*, Signal Process. **192** (2022), 108333. https://app.dimensions.ai/details/publication/pub.1141936169
47. A. Derevianko and P. Loucka, Search for similarity transformation between image point clouds using geometric algebra for conics, *Modelling and Simulation for Autonomous Systems*, Springer, 2022, pp. 215–226. https://link.springer.com/chapter/10.1007/978-3-030-98260-7_13
48. H. Liu, X. Liu, Z. Zhang, M. Liang, and C. Zhang, *A novel strategy regarding geometric product for liquids discrimination based on THz reflection spectroscopy*, Spectrochim. Acta Part A Molec. Biomolec. Spectrosc. **274** (2022), 121104. https://app.dimensions.ai/details/publication/pub.1146004459
49. A. Calado, P. Roselli, V. Errico, N. Magrofuoco, J. Vanderdonckt, and G. Saggio, *A geometric model-based approach to hand gesture recognition*, IEEE Trans. Syst. Man Cybern. Syst. **52** (2022), no. 10, 6151–6161. https://app.dimensions.ai/details/publication/pub.1144480878
50. D. Hildenbrand and A. Rockwood, *Geometric algebra computing for computer graphics using GAALOP*, ACM SIGGRAPH 2022 Courses, 2022, pp. 1–173. https://dl.acm.org/doi/10.1145/3532720.3535655
51. E. Wieser and U. Song, *Formalizing geometric algebra in lean*, Adv. Appl. Clifford Algeb. **32** (2022), no. 3, 28.
52. G. Papagiannakis, M. Kamarianakis, A. Protopsaltis, D. Angelis, and P. Zikas, *Project elements: a computational entity-component-system in a scene-graph pythonic framework, for a neural, geometric computer graphics curriculum*, 2023. https://arxiv.org/abs/2302.07691
53. M. N. Gevorkyan, A. V. Demidova, T. R. Velieva, A. V. Korol'kova, and D. S. Kulyabov, *Analytical-numerical implementation of polyvector algebra in Julia*, Programm. Comput. Softw. **48** (2022), no. 1, 49–58, DOI 10.1134/s0361768822010054.
54. A. Soiguine, *Explaining some weird quantum mechanical features in geometric algebra formalism*, Global J. Sci. Front. Res. **22** (2022), no. F3, 37–50, DOI 10.34257/gjsfrfvol22is3pg37.
55. J. Hrdina, A. Návrat, and P. Vašík, *Quantum computing based on complex Clifford algebras*, Quantum Inform. Process. **21** (2022), no. 9, 310.
56. J. Hrdina, D. Hildenbrand, A. Návrat, C. Steinmetz, R. Alves, C. C. Lavor, P. Vašík, and I. Eryganov, *Quantum register algebra: the mathematical language for quantum computing*, 2022. https://arxiv.org/abs/2208.02608
57. A. Soiguine, *The geometric algebra lift of qubits and beyond*, Authorea (2022). https://www.authorea.com/users/454445/articles/552055
58. P. Zanardi, *Quantum scrambling of observable algebras*, Quantum **6** (2022), 666, DOI 10.22331/q-2022-03-11-666.
59. W. Luo, Y. Wang, X. Zhang, D. Li, Z. Yu, Z. Yan, and L. Yuan, *Multilevel declassification method for geographic vector field data: a geometric algebra approach*, Adv. Appl. Clifford Algeb. **32** (2022), no. 5, 52.
60. X. Jiang, Y. Huang, and F. Zhang, *Study on spatial geometric similarity based on conformal geometric algebra*, Int. J. Environment. Res. Public Health **19** (2022), no. 17, 10807.
61. G. Lv, Z. Yu, L. Yuan, M. Wu, L. Zhou, W. Luo, and X. Zhang, Classification and description of geographic information: a comprehensive expression framework, *New thinking in GIScience*, Springer, 2022, pp. 75–83.
62. L. Wu, D. Wang, S. Feng, K. Song, Y. Zhang, and G. Yu, *Database systems for advanced applications, 27th International Conference, DASFAA 2022, Virtual Event, April 11-14, 2022, Proceedings, Part II*, Lecture Notes in Computer Science (2022), LNCS 12681, 256–263.
63. C. Xu, M. Nayyeri, Y.-Y. Chen, and J. Lehmann, *Geometric algebra based embeddings for static and temporal knowledge graph completion*, IEEE Trans. Knowl. Data Eng. **PP** (2022), no. 99, 1–1.
64. D. W. H. A. Silva, L. Harmon, and G. Delavignette, *Threshold secret sharing with geometric algebra*, Authorea (2022). https://www.authorea.com/users/459269/articles/555588
65. T. K. Tanev and A. Lekova, *Implementation of actors' emotional talent into social robots through capture of human head's motion and basic expression*, Int. J. Social Robot. **14** (2022), no. 7, 1749–1766, DOI 10.1007/s12369-022-00910-0.
66. Y. Li, Y. Wang, Y. Wang, C. Qian, and R. Wang, *Geometric algebra based recurrent neural network for multi-dimensional time-series prediction*, Front. Comput. Neurosci. **16** (2022), 1078150.
67. Y. Li, Y. Wang, R. Wang, Y. Wang, K. Wang, X. Wang, W. Cao, and W. Xiang, *GA-CNN: convolutional neural network based on geometric algebra for hyperspectral image classification*, IEEE Trans. Geosci. Remote Sens. **60** (2022), 1–14.
68. Q. Liu and W. Cao, *Geometric algebra graph neural network for cross-domain few-shot classification*, Appl. Intell. **52** (2022), no. 11, 12422–12435.
69. E. Grassucci, A. Zhang, and D. Comminiello, *PHNNS: lightweight neural networks via parameterized hypercomplex convolutions*, IEEE Trans. Neural Netw. Learn. Syst. (2022), 1–13.
70. M. Kamarianakis, I. Chrysovergis, N. Lydatakis, M. Kentros, and G. Papagiannakis, *Less is more: efficient networked VR transformation handling using geometric algebra*, Adv. Appl. Clifford Algeb. **33** (2023), no. 1, 6.
71. A. Pepe, J. Lasenby, and P. Chacón, *Learning rotations*, Math. Methods Appl. Sci. (2022), DOI doi/10.1002/mma.8698.
72. D. Ruhe, J. K. Gupta, S. de Keninck, M. Welling, and J. Brandstetter, *Geometric Clifford algebra networks*, 2023.
73. D. Zang, Y. Ding, X. Qu, X. Chen, K. Tang, and J. Zhang, *Traffic data prediction with geometric algebra spatial-temporal attention neural network*, 2022 7th International Conference on Big Data and Computing, 2022, pp. 14–19.

74. D. Zang, X. Chen, J. Lei, Z. Wang, J. Zhang, J. Cheng, and K. Tang, *A multi–channel geometric algebra residual network for traffic data prediction*, IET Intell. Transport Syst. **16** (2022), no. 11, 1549–1560.
75. D. Zang, Y. Ding, X. Qu, C. Miao, X. Chen, J. Zhang, and K. Tang, *Traffic-data recovery using geometric-algebra-based generative adversarial network*, Sensors (Basel, Switzerland) **22** (2022), no. 7, 2744.
76. C. Miao, W. Su, Y. Fu, X. Chen, and D. Zang, *Long-term traffic speed prediction based on geometric algebra ConvLSTM and graph attention*, 2022 IEEE International Conference on Smart Internet of Things (SmartIoT), 2022, pp. 108–115.
77. J. A. García-Limón, J. P. S. Rubio, R. Herrera-Guzmán, L. M. Rodriguez-Vidal, and C. M. Hernández-Mendoza, *Advances in computational intelligence, 21st Mexican International Conference on Artificial Intelligence, MICAI 2022, Monterrey, Mexico, October 24–29, 2022, Proceedings, Part I*, Lect. Notes Comput. Sci. (2022), LNCS 13612, 38–51.
78. Y. Oktar and M. Turkan, *Preserving spatio-temporal information in machine learning: a shift-invariant k-means perspective*, J. Signal Process. Syst. **94** (2022), no. 12, 1471–1483.
79. Y. Tu, S. Lin, J. Qiao, Y. Zhuang, and P. Zhang, *Alzheimer's disease diagnosis via multimodal feature fusion*, Comput. Biol. Medicine **148** (2022), 105901.
80. R. Wang, N. Fang, Y. He, Y. Li, W. Cao, and H. Wang, *Multi-modal medical image fusion based on geometric algebra discrete cosine transform*, Adv. Appl. Clifford Algeb. **32** (2022), no. 2, 19.
81. Y. Li, N. Fang, H. Wang, and R. Wang, *Multi-modal medical image fusion with geometric algebra based sparse representation*, Front. Gen. **13** (2022), 927222.
82. P. Zikas, A. Protopsaltis, N. Lydatakis, M. Kentros, S. Geronikolakis, S. Kateros, M. Kamarianakis, G. Evangelou, A. Filippidis, E. Grigoriou, D. Angelis, M. Tamiolakis, M. Dodis, G. Kokiadis, J. Petropoulos, M. Pateraki, and G. Papagiannakis, *Mages 4.0: accelerating the world's transition to VR training and democratizing the authoring of the medical metaverse*, IEEE Comput. Graph. Appl. **43** (2023), no. 2, 1–16.
83. G. Yao, A. Aierken, T. Li, X. Cao, S. Apaer, N. Anweier, J. Wu, X. Fulati, Y. F. Zhang, A. Tuerxunmaimaiti, and T. Tuxun, *The influence of geometric algebra in surgical practice of sleeve gastrectomy-single center experience*, Medicine **101** (2022), no. 43, e30783, DOI 10.1097/md.0000000000030783.
84. S. Jensen and J. Poling, *Teaching rotational physics with bivectors*, 2022. https://arxiv.org/abs/2207.03560
85. S. D. Brechet, *Rotations in classical mechanics using geometric algebra*, 2022. https://arxiv.org/abs/2210.16803
86. H. Ikemori, S. Kitakado, Y. Matsui, and T. Sato, *Runge-Lenz vector as a 3D projection of* $SO(4)$ *moment map in* $\mathbb{R}^4 \times \mathbb{R}^4$ *phase space*, 2022. https://arxiv.org/abs/2212.04931
87. S. D. Brechet, *Electrodynamics in geometric algebra*, 2022. https://arxiv.org/abs/2210.05601v1
88. D. Sen, *Electromagnetism according to geometric algebra: an appropriate and comprehensive formulation*, Pramana **96** (2022), no. 4, 165. https://app.dimensions.ai/details/publication/pub.1150794082
89. B. Wu, *A signature invariant geometric algebra framework for spacetime physics and its applications in relativistic dynamics of a massive particle and gyroscopic precession*, Scientif. Rep. **12** (2022), no. 1, 3981. https://www.nature.com/articles/s41598-022-06895-0.pdf
90. J. Xu, *Gravitational waves in gauge theory gravity with a negative cosmological constant*, Class. Quant. Grav. **39** (2022), no. 17, 175005. https://arxiv.org/pdf/2012.15001
91. A. J. S. Hamilton, *The supergeometric algebra*, Adv. Appl. Clifford Algeb. **33** (2023), no. 1, 12, DOI 10.1007/s00006-022-01256-6.
92. A. Lasenby, *Some recent results for* $SU(3)$ *and octonions within the geometric algebra approach to the fundamental forces of nature*, Math. Methods Appl. Sci. (2023). https://arxiv.org/abs/2202.06733
93. V. Zatloukal, *Real spinors and real Dirac equation*, Adv. Appl. Clifford Algeb. **32** (2022), no. 4, 45. https://arxiv.org/pdf/1908.04590
94. D. W. Marks, *Binary encoded recursive generation of quantum space-times*, Adv. Appl. Clifford Algeb. **32** (2022), no. 4, 51. https://link.springer.com/content/pdf/10.1007/s00006-022-01235-x.pdf
95. S. Alexander, *The 3-sphere instead of Hilbert space*, J. Appl. Math. Phys. **10** (2022), no. 09, 2733–2742. https://www.scirp.org/journal/PaperDownload.aspx?paperID=120000
96. A. Soiguine, *Explaining double split experiment with geometrical algebra formalism*, Appl. Sci. Innov. Res. **6** (2022), no. 1, p46. https://www.scholink.org/ojs/index.php/asir/article/download/4528/5198
97. S. Andoni, *Spin 1/2 one- and two-particle systems in physical space without eigen-algebra or tensor product*, Math. Methods Appl. Sci. (2022). https://arxiv.org/pdf/2212.09463
98. E. Trell, *A bottom-up 'Game of Lie' cellular automaton evolution in SO(3) root space both nucleating, crystallizing and space-filling the complete atomic realm*, J. Phys. Conf. Ser. **2197** (2022), no. 1, 12025, DOI 10.1088/1742-6596/2197/1/012025.
99. J. Christian, *Local origins of quantum correlations rooted in geometric algebra*, 2022. https://arxiv.org/abs/2205.11372
100. J. Böhm, J. L. Jacobsen, Y. Jiang, and Y. Zhang, *Geometric algebra and algebraic geometry of loop and Potts models*, J. High Energy Phys. **2022** (2022), no. 5, 68, DOI 10.1007/jhep05(2022)068.
101. A. H. Eid and F. G. Montoya, *A geometric procedure for computing differential characteristics of multi-phase electrical signals using geometric algebra*, 2022. https://arxiv.org/abs/2208.05917v1
102. A. H. Eid and F. G. Montoya, *A systematic and comprehensive geometric framework for multiphase power systems analysis and computing in time domain*, IEEE Access **10** (2022), 132725–132741. https://ieeexplore.ieee.org/ielx7/6287639/6514899/09992202.pdf
103. F. G. Montoya, F. M. Arrabal-Campos, A. Alcayde, X. Prado-Orbán, and J. Mira, *Geometric power and Poynting vector: a physical derivation for harmonic power flow using geometric algebra*, 2022 20th International Conference on Harmonics & Quality of Power (ICHQP), 2022, pp. 1–6. https://app.dimensions.ai/details/publication/pub.1149159268
104. F. G. Montoya, R. Baños, A. Alcayde, F. M. Arrabal-Campos, and J. Roldán-Pérez, *Geometric algebra applied to multiphase electrical circuits in mixed time-frequency domain by means of hypercomplex Hilbert transform*, Mathematics **10** (2022), no. 9, 1419. https://www.mdpi.com/2227-7390/10/9/1419/pdf?version=1650872705

105. M. Castro-Núñez, D. Londoño-Monsalve, and R. Castro-Puche, *M, the power definition in geometric algebra that unveils the shortcomings of the nonsinusoidal apparent power S*, Adv. Appl. Clifford Algeb. **32** (2022), no. 2, 18. https://app.dimensions.ai/details/publication/pub.1145698659
106. N. Sundriyal, J. M. Ramirez, and E. B. Corrachono, *A comparison of geometric algebra and harmonic domain for linear circuit analysis*, Authorea (2022). https://www.authorea.com/doi/pdf/10.22541/au.164848881.10317415
107. B. Cross, R. J. Cripps, and G. Mullineux, *$C^1$ and $G^1$ continuous rational motions using a conformal geometric algebra*, J. Comput. Appl. Math. **412** (2022), 114280. https://app.dimensions.ai/details/publication/pub.1146677761
108. J. A. Medrano-Hermosillo, R. Lozoya-Ponce, J. Ramírez-Quintana, and R. Baray-Arana, *Forward kinematics analysis of 6-DoF articulated robot using screw theory and geometric algebra*, 2022 XXIV Robotics Mexican Congress (COMRob), 2022, pp. 1–6. https://app.dimensions.ai/details/publication/pub.1153210017
109. E. Macias-Garcia, J. Zamora-Esquivel, N. Chel-Puc, and E. Bayro-Corrochano, *Iterative inverse kinematics based on screw rotors and the extended Kalman filter*, 2022 Asia Conference on Advanced Robotics, Automation, and Control Engineering (ARACE), 2022, pp. 81–86. https://app.dimensions.ai/details/publication/pub.1152875332
110. L. Lechuga-Gutierrez, E. Macias-Garcia, G. Martínez-Terán, J. Zamora-Esquivel, and E. Bayro-Corrochano, *Iterative inverse kinematics for robot manipulators using quaternion algebra and conformal geometric algebra*, Meccanica **57** (2022), no. 6, 1413–1428. https://app.dimensions.ai/details/publication/pub.1147080269
111. L. Wang, G. Yu, L. Sun, Y. Zhou, and C. Wu, *Motion generation of a planar 3R serial chain based on conformal geometric algebra with applications to planar linkages*, Mech. Sci. **13** (2022), no. 1, 275–290. https://ms.copernicus.org/articles/13/275/2022/ms-13-275-2022.pdf
112. G. Zhu, S. Wei, Y. Zhang, and Q. Liao, *CGA-based novel modeling method for solving the forward displacement analysis of 3-RPR planar parallel mechanism*, Mechan. Machine Theory **168** (2022), 104595. https://app.dimensions.ai/details/publication/pub.1142497725
113. G. Li, Y. Li, S. Weng, H. Wan, and D. Luo, *Design and simulation of a novel 6-DOF hybrid mechanism motion platform for pose adjustment of heavy equipment*, Iranian J. Sci. Technol. Trans. Mechan. Eng. **47** (2022), 1055–1078. https://app.dimensions.ai/details/publication/pub.1153631267
114. Z. Tong, G. Yu, J. Liu, and X. Zhao, *Research and application of spherical 2R mechanism based on conformal geometric algebra*, Res. Square (2022). https://www.researchsquare.com/article/rs-657363/latest.pdf
115. L. Xu, W. Ye, and Q. Li, *Design, analysis, and experiment of a new parallel manipulator with two rotational and one translational motion*, Mechan. Machine Theory **177** (2022), 105064. https://app.dimensions.ai/details/publication/pub.1150403980
116. Z. Song, S. Jiang, B. Chen, L. Wang, and H. Wu, *Analysis of dynamic modeling and solution of 3-RPS parallel mechanism based on conformal geometric algebra*, Meccanica **57** (2022), no. 6, 1443–1455. https://app.dimensions.ai/details/publication/pub.1146890652
117. I. Zaplana, H. Hadfield, and J. Lasenby, *Singularities of serial robots: identification and distance computation using geometric algebra*, Mathematics **10** (2022), no. 12, 2068. https://www.mdpi.com/2227-7390/10/12/2068/pdf?version=1655287883
118. M. Stodola and S. Frolík, *Self-calibration method of binocular vision based on conformal geometric algebra*, Math. Methods Appl. Sci. (2022), DOI 10.22541/au.163977066.68521892/v1.
119. I. Zaplana, H. Hadfield, and J. Lasenby, *Closed-form solutions for the inverse kinematics of serial robots using conformal geometric algebra*, Mechan. Machine Theory **173** (2022), 104835. https://lirias.kuleuven.be/bitstream/20.500.12942/695623/2/IK%2bGA.pdf
120. B. Kalkan, Z. Li, H.-P. Schröcker, and J. Siegele, *The study variety of conformal kinematics*, Adv. Appl. Clifford Algeb. **32** (2022), no. 4, 44. https://link.springer.com/content/pdf/10.1007/s00006-022-01227-x.pdf
121. J. Hrdina, A. Návrat, P. Vašík, and L. Zalabová, *A note on geometric algebras and control problems with SO(3)-symmetries*, Math. Methods Appl. Sci. (2022). https://arxiv.org/pdf/2111.11088
122. E. Bayro-Corrochano, J. Medrano-Hermosillo, G. Osuna-González, and U. Uriostegui-Legorreta, *Newton–Euler modeling and Hamiltonians for robot control in the geometric algebra*, Robotica **40** (2022), no. 11, 4031–4055. https://app.dimensions.ai/details/publication/pub.1149689719
123. Y. Chen, P. Huang, and Y. Ding, *An analytical method for corner smoothing of five-axis linear paths using the conformal geometric algebra*, Comput.-Aided Des. **153** (2022), 103408. https://app.dimensions.ai/details/publication/pub.1150929501
124. B. Navrátilová, R. Byrtus, and M. Holub, *Geometric algebra methods in volumetric accuracy analysis*, Math. Methods Appl. Sci. (2022), DOI 10.1002/mma.8494.
125. T. Uzunović, F. G. Montoya, A. Osmanović, F. M. Arrabal-Campos, A. Alcayde, A. H. Eid, and A. Šabanović, *Combining real-time parameter identification and robust control algorithms for effective control of electrical machines*, 2022 International Conference on Electrical Machines (ICEM), 2022, pp. 2391–2396. https://app.dimensions.ai/details/publication/pub.1151864205
126. S. Vitabile, S. Franchini, and G. Vassallo, *An optimized architecture for CGA operations and its application to a simulated robotic arm*, Electronics **11** (2022), no. 21, 3508. https://www.mdpi.com/2079-9292/11/21/3508/pdf?version=1667888434
127. H. S. Ramírez, M. A. Aguilar-Orduña, and B. C. Gómez-León, *The invariance control approach to sliding regimes in switched systems: a geometric algebra viewpoint*, 2022 16th International Workshop on Variable Structure Systems (VSS), 2022, pp. 29–34. https://app.dimensions.ai/details/publication/pub.1151456976
128. H. S. Ramírez, M. A. Aguilar-Orduña, and B. C. Gómez-León, *Sliding mode control of switched systems: a geometric algebra approach*, Asian J. Control **25** (2022), no. 1, 15–25. https://app.dimensions.ai/details/publication/pub.1146121878
129. A. Derevianko and P. Vašík, *Solver-free optimal control for linear dynamical switched system by means of geometric algebra*, Math. Methods Appl. Sci. (2022), DOI 10.1002/mma.8752.
130. T. Löw and S. Calinon, *Geometric algebra for optimal control with applications in manipulation tasks*, 2022. arXiv, DOI 10.48550/arxiv.2212.07237.

## AUTHOR BIOGRAPHIES

**Eckhard Hitzer** holds a PhD in theoretical physics from the University of Konstanz in Konstanz, Germany. He is Senior Associate Professor at College of Liberal Arts of International Christian University in Tokyo, Japan. His main research interests are theory and application of Clifford geometric algebras. He is a Visiting Fellow for Religious, Philosophical and Scientific Foundations of Sustainable Society at Europa Institute of Sophia University, Tokyo, Japan. He recently published *Quaternion and Clifford Fourier Transforms*, Chapman and Hall/CRC, 474 pp., Sep. 2021. Editor of GA-Net, and GA-Net Updates (blog).

**Dr. Manos Kamarianakis** is a mathematician researcher that specializes in Computational Geometry and Computer Graphics. He holds a PhD in Computational Geometry, a M.Sc. in Mathematics (specializing in design of algebra-based algorithms), and two B.S. (pure and applied mathematics) from University of Crete. Throughout his studies, he was awarded multiple scholarships, including a Onassis Foundation scholarship. He was, until recently, a post-doctoral researcher in the Department of Mathematics and Applied Mathematics, UoC, focusing on research topics regarding Algebra, Logic and Number theory. For the past 3.5 years, he is a member of the Foundation for Research and Technology Hellas-Institute of Computer Science (FORTH-ICS) spinoff startup ORamaVR, where his role as the R&D director involves the design of innovative, geometric-based algorithms, suitable for Augmented, Virtual and Mixed Reality. Manos is also a visiting researcher of FORTH-ICS and specifically the Human–Computer Interaction Lab as well as also a temp professor at University of Crete, teaching Computer Graphics. He has published multiple articles on these topics and recently, his paper *Never Drop the ball* received the Best Paper and Presentation Award from the CGI2021 Conference (ENGAGE Workshop).

**Dr. George Papagiannakis** co-founder and CEO of ORamaVR, is a computer scientist specialized in computer graphics systems, extended reality algorithms and geometric computational models. His academic credentials include serving as Professor of Computer Graphics at the Computer Science department of the University of Crete, Greece, as Affiliated Research Fellow at the Human Computer Interaction Laboratory of the Institute of Computer Science in the Foundation for Research and Technology Hellas, Heraklion, Greece, where he leads the CG Group and as visiting Prof of CS at the University of Geneva. He has more than 100 publications in the field, and he is a member of CGS (Board Member), IEEE, Eurographics, ACM and SIGGRAPH professional societies. In 2011 he was awarded a Marie-Curie Intra-European Fellowship for Career Development from the European Commission's Research Executive Agency. He was conference chair of the Computer Graphics International 2016 Conference, in cooperation with CGS, ACM, ACM SIGGRAPH and Eurographics Associations. In 2017 he published a Springer-Nature book on Mixed Reality and Gamification which achieved more than 77.000 downloads so far.

**Petr Vašík** is a mathematician at the Faculty of Mechanical Engineering of Brno University of Technology, Brno, Czech Republic. He is an associated professor and the Head of the Institute of Mathematics. His main research areas are differential geometry, particularly geometric control theory, and geometric algebras with the focus on applications in engineering. He is the organizer of several series of international conferences, such as Applied Geometric Algebras in Computer Science and Engineering (AGACSE 2021, main organizer), Empowering Novel Geometric Algebra for Graphics & Engineering (ENGAGE 2022), and Modeling & Simulation for Autonomous Systems. He is a guest editor of two special issues in Mathematical Methods in the Applied Sciences, Wiley.

**How to cite this article:** E. Hitzer, M. Kamarianakis, G. Papagiannakis, and P. Vašík, *Survey of new applications of geometric algebra*, Math. Meth. Appl. Sci. (2023), 1–17, DOI 10.1002/mma.9575.